Electromagnetic Theory — Module 2

KTU Engineering Physics | Dynamic EM Fields, Maxwell's Equations, Wave Equations, Boundary Conditions

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1. Electromagnetic Fields — Introduction

A field consisting of both electric and magnetic components is called an **electromagnetic (EM) field**. In *static* EM fields, the two components are independent of each other. In *dynamic* (time-varying) EM fields they are interdependent.

Sources of EM Fields

Stationary charges
→ Electrostatic fields

Steady (DC) currents
→ Magnetostatic fields

Time-varying currents
→ EM fields / waves

Accelerated charges
→ Radiation (time-varying fields)

Types of Fields

- **Electrostatic fields** — produced by static electric charges.
- **Magnetostatic fields** — due to motion of charges at uniform velocity (DC) or static magnetic charges (poles).
- **Time-varying fields / waves** — due to accelerated charges or time-varying currents. Any pulsating current produces radiation.

Time-varying E and H fields are represented as **E(x, y, z, t)** and **H(x, y, z, t)**.

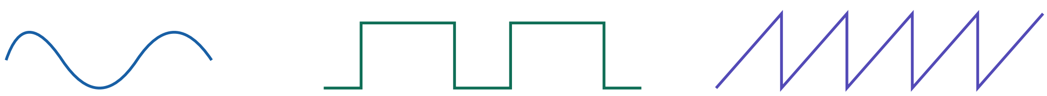

Fig: Examples of time-varying currents that produce EM waves

2. Maxwell's Equations

Maxwell's equations are based on three fundamental laws:

Ampere's Law | Faraday's Law | Gauss's Law

Derivation Overview

From Gauss's Law: Electric flux $\psi = \oint_S D \cdot ds = Q_{enc} = \int_V \rho_v \, dv$

Applying the divergence theorem and comparing:

$$\nabla \cdot \mathbf{D} = \rho_v$$

Point / Differential form — Maxwell's 1st equation

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v \, dv$$

Integral form

Similarly (Gauss's Law for Magnetism):

$\nabla \cdot \mathbf{B} = 0$ → Differential form

$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ → Integral form (no magnetic monopoles)

From Faraday's Law

Oersted (1820) discovered steady current → magnetic field. Faraday discovered a time-varying magnetic field → electric current.

Faraday's Law: A time-varying magnetic field induces a voltage (EMF) equal to the time rate of change of magnetic flux linkage.

$$V_{emf} = -d\lambda/dt = -N\, d\Phi_m/dt$$

General (N turns)

$$V_{emf} = \oint_L \mathbf{E}\cdot d\mathbf{l} = -\int_S (\partial \mathbf{B}/\partial t)\cdot d\mathbf{s}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

Differential form — Maxwell's 2nd equation

From Ampere's Circuit Law (Maxwell's Modification)

Original Ampere's law: $\nabla \times \mathbf{H} = \mathbf{J}$. But for time-varying conditions, this leads to a contradiction with the continuity equation ($\nabla\cdot\mathbf{J} = -\partial\rho_v/\partial t \neq 0$).

Maxwell introduced the **displacement current density $J_d = \partial D/\partial t$** to resolve this:

> **Displacement current** $I_d = \int_S \mathbf{J}_d\cdot d\mathbf{s} = \int_S (\partial \mathbf{D}/\partial t)\cdot d\mathbf{s}$
> This is different from conduction current density **J** (= σ**E**).

$$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$$

Differential form — Maxwell's 3rd equation (modified Ampere)

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = I_{enc} = \int_S \mathbf{J}\cdot d\mathbf{s} + \int_S (\partial \mathbf{D}/\partial t)\cdot d\mathbf{s}$$

Integral form

Summary of Maxwell's Equations

Law	Differential Form	Integral Form
Gauss's Law (E)	$\nabla\cdot\mathbf{D} = \rho_v$	$\oint \mathbf{D}\cdot d\mathbf{s} = \int \rho_v\, dv$
Gauss's Law (B)	$\nabla\cdot\mathbf{B} = 0$	$\oint \mathbf{B}\cdot d\mathbf{s} = 0$
Faraday's Law	$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$	$\oint \mathbf{E}\cdot d\mathbf{l} = -\int (\partial \mathbf{B}/\partial t)\cdot d\mathbf{s}$
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$	$\oint \mathbf{H}\cdot d\mathbf{l} = \int \mathbf{J}\cdot d\mathbf{s} + \int (\partial \mathbf{D}/\partial t)\cdot d\mathbf{s}$

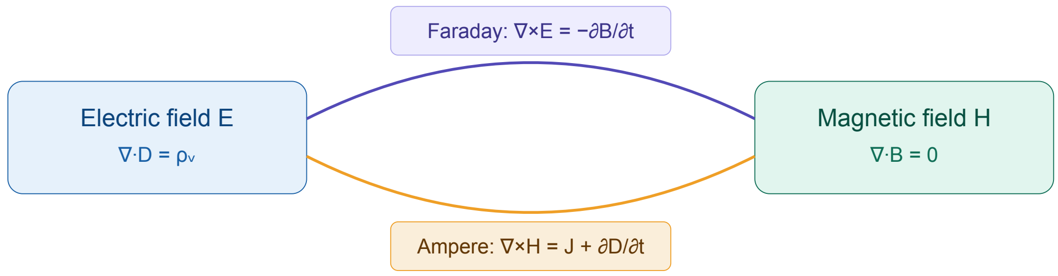

Fig: Interdependence of E and H in dynamic EM fields

3. Wave Equations — Free Space / Lossless Medium

Waves are means of transporting energy or information. EM wave examples: radio waves, TV signals, light rays.

Media Classification

Medium	σ	ε	μ
Free space	0	ε_0	μ_0
Lossless dielectric	0	$\varepsilon_0\varepsilon_r$	$\mu_0\mu_r$
Lossy dielectric	≠0	$\varepsilon_0\varepsilon_r$	$\mu_0\mu_r$
Good conductor	→∞	ε_0	$\mu_0\mu_r$

Derivation for Free Space ($\rho_v = 0, \sigma = 0$)

Simplified Maxwell's equations for free space: $\mathbf{D} = \varepsilon_0\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$, $\mathbf{J} = 0$

> Starting from $\nabla\times\mathbf{H} = \varepsilon_0\, \partial\mathbf{E}/\partial t$ and applying curl, then using $\nabla\times\mathbf{E} = -\mu_0\, \partial\mathbf{H}/\partial t$:

$$\nabla^2\mathbf{H} = \mu_0\varepsilon_0\, \partial^2\mathbf{H}/\partial t^2$$

Wave equation for H — free space

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0\, \partial^2\mathbf{E}/\partial t^2$$

Wave equation for E — free space

These are called the **wave equations**. A wave is a function of both space and time. The term $\mu_0\varepsilon_0 = 1/c^2$, where c is the speed of light.

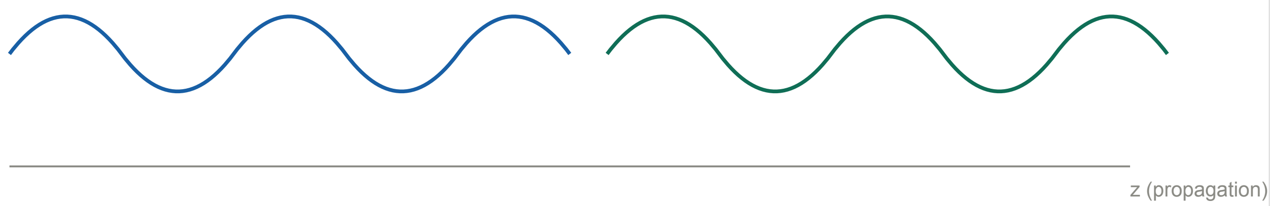

Fig: E and H fields in a plane EM wave (both transverse, mutually perpendicular)

4. Wave Equations — Conducting Medium

For a homogeneous, isotropic, linear medium with conductivity σ, charge density $\rho_v = 0$ (no net charge inside conductor).

The conduction current density: $\mathbf{J} = \sigma\mathbf{E}$

From Maxwell's modified Ampere's law: $\nabla\times\mathbf{H} = \sigma\mathbf{E} + \varepsilon(\partial\mathbf{E}/\partial t)$

Taking curl and applying vector identity $\nabla\times\nabla\times\mathbf{H} = \nabla(\nabla\cdot\mathbf{H}) - \nabla^2\mathbf{H}$:

$$\nabla^2\mathbf{H} = \mu\sigma\,(\partial\mathbf{H}/\partial t) + \mu\varepsilon\,(\partial^2\mathbf{H}/\partial t^2)$$

Wave equation for H — conducting medium

$$\nabla^2\mathbf{E} = \mu\sigma\,(\partial\mathbf{E}/\partial t) + \mu\varepsilon\,(\partial^2\mathbf{E}/\partial t^2)$$

Wave equation for E — conducting medium

Note: The extra term $\mu\sigma(\partial/\partial t)$ accounts for *ohmic losses* in the conductor. For free space ($\sigma = 0$), these reduce to the lossless wave equations.

Parameter	Free Space	Conducting Medium
Wave equation (E)	$\nabla^2 E = \mu_0\varepsilon_0\, \partial^2 E/\partial t^2$	$\nabla^2 E = \mu\sigma\, \partial E/\partial t + \mu\varepsilon\, \partial^2 E/\partial t^2$

Parameter	Free Space	Conducting Medium
Conductivity σ	0	≠ 0
Attenuation	None	Present (exponential decay)

5. Boundary Conditions

When an electric/magnetic field exists in a region of two different media, conditions the field must satisfy at the interface are called **boundary conditions**.

Why They Matter

If the field on one side of a boundary is known, boundary conditions help determine the field on the other side.

Three types of interfaces:

Dielectric–Dielectric | Conductor–Dielectric | Conductor–Free Space

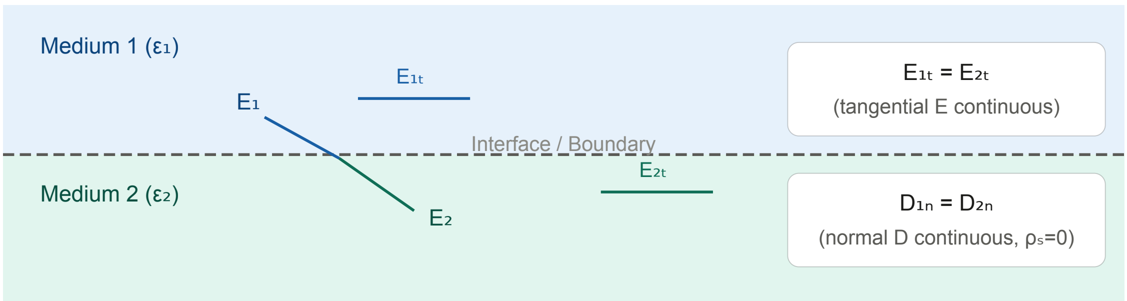

Fig: Dielectric–Dielectric boundary — E field decomposition into tangential and normal components

Dielectric–Dielectric Boundary Conditions (Summary)

Tangential E: $E_{1t} = E_{2t}$ → $D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$ (discontinuous)
Tangential E is *continuous* across the boundary.

Normal D: $D_{1n} - D_{2n} = \rho_s$ (if $\rho_s = 0$) $D_{1n} = D_{2n}$
$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$ → $E_{1n}/E_{2n} = \varepsilon_2/\varepsilon_1$ (discontinuous)

Conductor–Dielectric Boundary Conditions

Inside a perfect conductor ($\sigma = \infty$): **E = 0**

Tangential: $E_t = 0$, $D_t = \varepsilon E_t = 0$
Normal: $D_n = \rho_s$, $\varepsilon_0 E_n = \rho_s$

Summary: No electric field exists within a conductor under static conditions.

Magnetic Boundary Conditions

Component	Condition	Behaviour
Normal B	$B_{1n} = B_{2n}$	Continuous
Normal H	$\mu_1 H_{1n} = \mu_2 H_{2n}$	Discontinuous
Tangential H (k=0)	$H_{1t} = H_{2t}$	Continuous
Tangential B (k=0)	$B_{1t}/\mu_1 = B_{2t}/\mu_2$	Discontinuous

6. Capacitance

Capacitance C = Q/V, where V is the potential difference due to equal and opposite charges of magnitude Q.

Capacitance of a Co-axial Cable

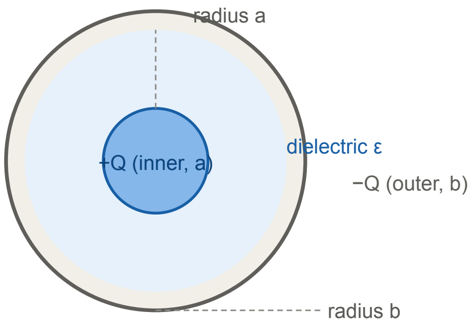

Fig: Cross-section of a co-axial cable (inner radius a, outer radius b)

Electric field between conductors: $\mathbf{E} = \rho_L / (2\pi\varepsilon\rho)$

Potential difference $V = (\rho_L/2\pi\varepsilon)\ln(b/a)$

$$C = 2\pi\varepsilon / \ln(b/a) \quad \text{F/m}$$

Capacitance per unit length of coaxial cable

Capacitance of Two-Wire Transmission Line

Two parallel wires A and B of radius r, separation D ($D \gg r$), carrying +Q and −Q per unit length:

$$C = \pi\varepsilon_0 / \ln(D/r) \quad \text{F/m}$$

Capacitance per unit length of two-wire line

7. Inductance

A circuit carrying current I produces a magnetic field B that causes flux $\Psi_m = \int B\cdot ds$ to pass through each turn.

Flux linkage $\lambda = N\Psi_m$, and $\lambda \propto I \rightarrow \lambda = LI$

$$L = \lambda/I = N\Psi_m/I \quad \text{[Henry (H)]}$$

Inductance definition

Inductance of a Co-axial Line

Consider a coaxial line with inner radius a and outer radius b, carrying current I on the inner conductor.

Magnetic flux density between conductors: $\mathbf{B = \mu I/(2\pi\rho)}$

Total flux linkage for length l: $\Psi_m = (\mu I l/2\pi)\ln(b/a)$

$$L = \mu \ln(b/a) / (2\pi) \quad \text{H/m}$$

Inductance per unit length of coaxial cable

Inductance of Two-Wire Transmission Line

$$L = \mu \ln(D/r) / \pi \quad \text{H/m}$$

Inductance per unit length of two-wire transmission line

8. Energy Stored in Magnetostatic Field

The magnetic energy in the field of an inductor: $W_M = \frac{1}{2}LI^2$

Expressing in terms of B or H by considering a differential volume element with inductance ΔL:

$$\Delta W_M = \tfrac{1}{2}\,\mu H^2\,\Delta V = \tfrac{1}{2}\,B\cdot H\,\Delta V$$

$$W_M = \tfrac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}\,dV = \tfrac{1}{2}\int_V \mu H^2\,dV$$

Total magnetic energy stored

Energy density $w_m = \frac{1}{2}\mu H^2 = \frac{1}{2} B\cdot H$ [J/m³]

9. Magnetic Scalar and Vector Potentials

Analogous to the electric potential V (where $E = -\nabla V$), a potential associated with magnetostatic fields can be defined.

Scalar Magnetic Potential V_m

$$\mathbf{H} = -\nabla V_m$$

Valid only where **J** = 0 (current-free region). Satisfies Laplace's equation:

$$\nabla^2 V_m = 0$$

Vector Magnetic Potential A

$$\mathbf{B} = \nabla \times \mathbf{A}$$

For line, surface, and volume currents:

$$A = \int \mu_0 I\,dl \,/\, (4\pi R)$$

Two key vector identities:

- $\nabla \times (\nabla V) = 0$ (curl of gradient is zero)
- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ (divergence of curl is zero)

Worked Example

Given: $A = -\rho^2/4\ a_z$ Wb/m. Find flux crossing $\varphi = \pi/2$, $1 \le \rho \le 2$m, $0 \le z \le 5$m.

$\mathbf{B} = \nabla \times \mathbf{A} = -(\partial A_z/\partial\rho)\ a_\varphi = \rho/2\ a_\varphi$

$\Psi_m = \int B \cdot ds = \frac{1}{2} \iint \rho\ d\rho\ dz = [\rho^2/4]_1^2 \times 5 = 15/4 =$ **3.75 Wb**